Math 255A' Lecture 27 Notes

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1 Riesz Functional Calculus and The Gelfand Transform

1.1 Riesz functional calculus

Theorem 1.1 (Cauchy's integral formula)**.** *Let $G \subseteq \mathbb{C}$ be open, let $f : G \to \mathbb{C}$ be holomorphic, and let $\Gamma = \gamma_1 \cup \cdots \cup \gamma_n$ be a system of contours such that the total winding number around any point in $\mathbb{C} \setminus G$ is 0. Let $z \in G \setminus \Gamma$. Then*

$$(\textit{winding \# of } \Gamma \textit{ around } z) f^{(k)}(z) = \frac{k!}{2\pi i} \oint_\Gamma \frac{1}{(z-w)^{k+1}} f(w)\, dw \qquad \forall k \geq 0.$$

This is ok when the target space is a Banach space X. The idea is that if $\mathscr{A}$ is a Banach algebra over $\mathbb{C}$ with identity and $a \in \mathscr{A}$, we let G be an open neighborhood of $\sigma(a)$. Then there exists some $\Gamma = \gamma_1 \cup \cdots\ cup \gamma_m$ such that the winding number is 0 around any point in $\mathbb{C} \setminus G$ and 1 around any point in $\sigma(a)$.

Now define

$$f(a) := \frac{1}{2\pi} \oint_\Gamma f(w) \cdot (a-w)^{-1}\, dw.$$

This is well-defined because if we define this with Γ and Γ', the difference is a sum of zeros by the Cauchy integral formula. Here are the properties of the functional calculus we get from this:

Proposition 1.1. *Let $\mathscr{A}$ be a Banach algebra with identity, let $a\ in \mathscr{A}$, and $\operatorname{Hol}(a)$ be the functions holomorphic on the spectrum of a. Then*

1. $\operatorname{Hol}(a) \to \mathscr{A} : f \mapsto f(a)$ *is an algebra homomorphism.*

2. *If $f(z) = \sum_{k \geq 0} \alpha_k z^k$ has radius of convergence $> r(a)$, then*

$$f(a) = \sum_k \alpha_k a^k$$

3. *If $f_1, f_2, \dots, f$ are all holomorphic on $G \supseteq \sigma(a)$ and $f_n \to f$ uniformly on compact subsets of G, then $f_n(a) \to f(a)$ in $\| \cdot \|_\infty$.*

Remark 1.1. These properties uniquely determine this algebra homomorphism. The proof uses Runge's theorem; first do this for rational functions, and then extend via density.

1.2 Abelian Banach algebras

To return to the spectral theorem, we first need some considerations about abelian Banach algebras.

Example 1.1. Let $C(X)$ be a compact Hausdorff space. Then $C(X)$ is an abelian, Banach algebra. The maximal ideals in $C(X)$ are the sets $\{f : f(x) = 0\}$. This tells you that you can recover X by looking at the maximal ideals of $C(X)$.

Theorem 1.2 (Gelfand-Mazur)**.** *Let $\mathscr{A}$ be a Banach algebra with identity. Assume that $\mathscr{A}$ is a division ring (every nonzero element of $\mathscr{A}$ is invertible). Then $\mathscr{A} = \mathbb{C} \cdot 1$.*

Remark 1.2. We are not assuming $\mathscr{A}$ is abelian, but this follows from the proof.

Proof. Let $a \in \mathscr{A}$, and choose $\lambda \in \sigma(a)$. Then $a - \lambda$ is not invertible, so $a - \lambda = 0$. So $a = \lambda 1$. □

Proposition 1.2. *Let $\mathscr{A}$ be a unital, abelian Banach algebra over $\mathbb{C}$ If $h : \mathscr{A} \to \mathbb{C}$ is a homomorphism (sending $1 \mapsto 1$), then $\ker h$ is a maximal ideal, and all maximals ideals arise this way uniquely.*

Proof. If $a \in \ker h$ and $b \in \mathscr{A}$, then $h(ab) = 0h(b) = 0$, so $ab \in \ker h$. So $\ker h$ is an ideal.

If $\ker H \subseteq M \subsetneq \mathscr{A}$, where M is an ideal, then $h(M)$ is a subspace of $\mathbb{C}$ (and actually an ideal). Then $h(M) = \mathbb{C}$ or $\{0\}$. Since M is proper, we get $h(M) = \{0\}$. So $M = \ker h$ is an ideal and is in fact maximal.

Now let M be a maximal ideal. There is the quotient map $Q : \mathscr{A} \to \mathscr{A}/M$. Since M is maximal, $\mathscr{A}$ has no nontrivial ideals. Then all nonzero elements are invertible, so by Gelfand-Mazur, we get that $\mathscr{A}/M = \mathbb{C}1_{\mathscr{A}/M}$. If we call the isomorphism $\pi : \mathscr{A}/M \to \mathbb{C}1_{\mathscr{A}/M}$, then $M = \ker(\pi \circ Q)$. □

Lemma 1.1.

$$\{a \in \mathscr{A} : a \textit{ not invertible}\} = \bigcup_{\substack{M \textit{max.} \\ \textit{ideal}}} M.$$

Proof. If a is in the left hand side, then $\{ab : b \in \mathscr{A}\}$ is an ideal without 1. So it is contained in a maximal ideal.

On the other hand, if $ab = 1$ and $a \in M$ for some ideal, then $1 \in M$. So $b = 1b \in M$, making $M = \mathscr{A}$. □

We take the convention that $\|1\|_{\mathscr{A}} = 1$.

Proposition 1.3. *Any homomorphism $h : \mathscr{A} \to \mathbb{C}$ is continuous with $\|h\|_{\mathscr{A}^*} = 1$.*

The idea is that homomorphisms are a special kind of linear functional, the ones that preserve multiplication. This says that they are all contained in the unit ball of $\mathscr{A}^*$.

Proof. h is continuous because $\ker h = M$ is closed. To show the norm estimate, we have $h(1) = 1$, which gives $\|h\|_{\mathscr{A}^*} \geq 1$. Now let $a \in \mathscr{A}$, and let $\lambda = h(a) \in \mathbb{C} \setminus \{0\}$. Then $h(1 - \frac{a}{\lambda}) = 0$, so $1 - \frac{a}{\lambda}$ is not invertible. Then $\|\frac{a}{\lambda}\| \geq 1$. This gives $|\lambda| \leq \|a\|$. $\square$

1.3 Maximal ideal spaces and the Gelfand transform

Definition 1.1. The **maximal ideal space** of $\mathscr{A}$ is

$$\Sigma = \{h \in \mathscr{A}^* : h \text{ unital homomorphism}\}.$$

Proposition 1.4. *Σ is compact for the weak* topology.*

Proof. $\Sigma \subseteq B_{\mathscr{A}^*}$, which is compact by Banach-Alaoglu. Also,

$$\Sigma = \{h \in B_{\mathscr{A}^*} : h(1) = 1\} \cap \bigcap_{a,b \in \mathscr{A}} \{h \in B_{\mathscr{A}^*} : h(ab) - h(a)h(b) = 0\},$$

which is an intersection of weak*-closed sets. So Σ is compact. $\square$

Theorem 1.3. *If X is a nonempty, compact, Hausdorff space, then $x \mapsto \delta_x$ is a homeomorphism $X \to \Sigma$, the maximal ideal space at $C(X)$.*

Proof. We only need to show that every maximal ideal M in $C(X)$ has the form $\{f : f(x) = 0\}$. By Riesz representation, $h(f) = \int_X f \, d\mu$. Since $\|h\| = 1$, $h(1) = 1$. So μ is a probability measure. Now $f \in M \iff \int f \, d\mu = 0$. And if $f \in M$, then $|f|^2 = f\overline{f} \in M$¡ so $\int |f|^2 \, d\mu = 0$. Check that this implies that the support of μ is a singleton. $\square$

Proposition 1.5. *If $a \in \mathscr{A}$, then $\sigma(a) = \{h(a) : h \in \Sigma\}$.*

Proof.

$$\begin{aligned} \lambda \in \sigma(a) &\iff a - \lambda \text{ not invertible} \\ &\iff a - \lambda \text{ is contained in some maximal } M \\ &\iff h(a) - \lambda = 0 \text{ for some } h \in \Sigma. \end{aligned}$$

$\square$

Definition 1.2. The **Gelfand transform** of $a \in \mathscr{A}$ is the function $\widehat{a} : \Sigma \to \mathbb{C}$ with $\widehat{a}(h) = h(a)$.

Now we can basically write the functional calculus but in reverse:

Theorem 1.4. *$a \mapsto \widehat{a}$ is a continuous homomorphism $\mathscr{A} \to C(\Sigma)$. Its kernel is* $\operatorname{rad}(A) = \bigcap_{h \in \Sigma} \ker h$*, and*

$$\|\widehat{a}\|_{\sup} = \lim_n \|a^n\|^{1/n} \leq \|a\|.$$

Proof. $\widehat{a} \in C(\Sigma)$ because $\widehat{a} = h(a)$ is the kind of functional which defines the weak* topology. The expression for the norms is the spectral radius formula. Lastly, $\widehat{a} = 0$ for all h if and only if $h \in \operatorname{rad} A$. This is an ideal (and intersection of ideals is an ideal). $\square$

The Gelfand transform is the canonical "best possible way to compare $\mathscr{A}$ to continuous functions on something." It's the best way because if we have another map $\mathscr{A} \to C(X)$, the radical will still get sent to 0. Next time, we will discuss conditions under which this map is surjective.

Example 1.2. Let $V \in \mathcal{B}(L^2([0,1]))$ be the Volterra operator, so $\sigma(V) = \{0\}$. Then $\|V^n\|^{1/n} \to 0$. Let $\mathscr{A} = \overline{\{p(V) : p \in \mathbb{C}[x]\}}$. This is an abelian Banach algebra with identity. The Gelfand transform sends $V \mapsto \widehat{V} = 0$. Then if $p(x) = \sum_{k=0}^{n} a_k x^k$, $\widehat{p(V)} = a_0 \cdot 1$. So the kernel of the Gelfand transform is the unique maximal ideal. You can check that this is $\overline{\{p(V) : p \in \mathbb{C}[X], p(0) = 0\}}$.